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TOWARDS A SOLUTION OF THE PROBLEM OF LOCAL EFFECTS ON CYLINDRICAL SHELLS-PANELS

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We investigate the stressed-deformed state of cylindrical shells-panels with a local normal load. The state is constructed by matching solutions for approximate differential equations of the theory of shells. Such a procedure, which received the name asymptotic synthesis and which demonstrated its effectiveness in considering closed shells, made it possible to obtain simple formulas for maximum stresses and normal displacement. These formulas are compared with the solution constructed in the present article on the basis of the Vlasov–Donnell equations.

Introduction. The calculation of cylindrical shells exposed to local force and temperature effects has received a sufficiently considerable amount of attention in the literature published at home and abroad. Significant results on this problem were obtained by P. P. Beilard [1] and V. M. Darevskii [2]. A considerable number of more recent publications of other authors are given in [3]. Below, to calculate a shell-panel, we resort to the method of asymptotic synthesis (MAS) of the stressed state, which was formulated in [4] and developed in [5] for the local stressed state of closed cylindrical shells.

1. Statement of the Problem, Solution of Resolving Equations. Let the dimension of a shell-panel along the generatrix be substantially larger than in the transverse direction. An external local normal load is distributed over the segment of the directional circumference (Fig. 1) and can be represented in the form

$$p(\alpha, \beta) = R^{-1} q(\beta)\, \delta(\alpha - 0)\,, \tag{1}$$

where

$$q(\beta) = \sum_{n=1}^{\infty} q_n \sin \gamma\beta\,, \quad \gamma = n\pi/\theta\,. \tag{2}$$

The second multiplier in Eq. (1) is the Dirac function. Taking into account its integral representation

$$\delta(\alpha - 0) = \pi^{-1} \int_0^{\infty} \cos \alpha\lambda d\lambda$$

and series (2), we write expression (1) as

$$p(\alpha, \beta) = (\pi R)^{-1} \sum_{n=1}^{\infty} q_n \sin \gamma\beta \int_0^{\infty} \cos \alpha\lambda d\lambda\,. \tag{3}$$

According to the method of asymptotic synthesis [4], the stressed state of the shell at low numbers of the harmonics n can be represented, without introducing a perceptible error, by the sum of the ground state and the edge effect. The first of these is described by the semimomentless theory of shells, whose resolving equation, with

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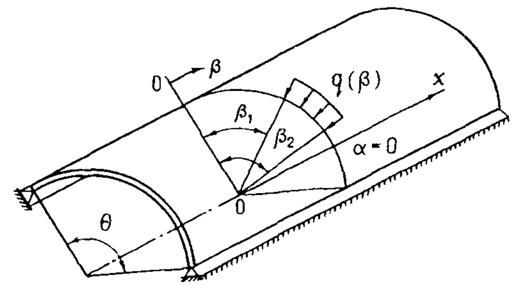

Fig. 1. Diagram of radial loading of shell-panel

the assumption of a strong inequality $|\partial^2\Phi/\partial\beta^2| >> |\Phi|$, changes to the following equation, called the Schorer two-term equation [6]:

$$\frac{\partial^4\Phi}{\partial\alpha^4} + c^2\frac{\partial^8\Phi}{\partial\beta^8} = \frac{R^2}{Eh}p(\alpha,\beta), \tag{4}$$

where $c^2 = h^2/(12(1-\nu^2)R^2)$; $\Phi(\alpha,\beta)$ is the resolving function.

Displacements, stresses, and bending moments are related to the resolving function by the following differential relationships:

$$w^o = \frac{\partial^4\Phi}{\partial\beta^4},\quad T_1^o = -\frac{Eh}{R}\frac{\partial^4\Phi}{\partial\alpha^2\partial\beta^2},\quad G_2^o = -\frac{D}{R^2}\frac{\partial^6\Phi}{\partial\beta^6},\quad G_1^o = \nu G_2^o,\quad D = Ehc^2R^2. \tag{5}$$

Using this variant of the theory of shells, Odquist [7] proposed a simple formula for calculating the normal displacement of a closed shell under the action of a concentrated force.

The solution of Eq. (4), damping out when $\alpha \to \pm\infty$, is the function

$$\Phi(\alpha,\beta) = \frac{R}{\pi Eh}\sum_{n=1}^{\infty} q_n \sin\gamma\beta \int_0^{\infty}\frac{\cos\alpha\lambda}{\lambda^4 + c^2\gamma^8}d\lambda. \tag{6}$$

According to Eqs. (5) and (6), for normal displacement and force factors we find

$$\begin{gathered} w^o = \frac{R}{\pi Eh}\sum_{n=1}^{\infty} q_n\gamma^4\sin\gamma\beta\int_0^{\infty}\frac{\cos\alpha\lambda}{\lambda^4+c^2\gamma^8}d\lambda, \\ T_1^o = -\frac{1}{\pi}\sum_{n=1}^{\infty} q_n\gamma^2\sin\gamma\beta\int_0^{\infty}\frac{\lambda^2\cos\alpha\lambda}{\lambda^4+c^2\gamma^8}d\lambda, \\ G_2^o = \frac{D}{\pi REh}\sum_{n=1}^{\infty} q_n\gamma^6\sin\gamma\beta\int_0^{\infty}\frac{\cos\alpha\lambda}{\lambda^4+c^2\gamma^8}d\lambda. \end{gathered} \tag{7}$$

We will confine ourselves to the analysis of these factors on the loading line $\alpha = 0$, where they attain the highest values. Considering that [8]

$$\int_0^{\infty}\frac{\lambda^2 d\lambda}{\lambda^4+c^2\gamma^8} = \frac{\pi\sqrt{2}}{4\sqrt{c}\,\gamma^2},\quad \int_0^{\infty}\frac{d\lambda}{\lambda^4+c^2\gamma^8} = \frac{\pi\sqrt{2}}{4c\sqrt{c}\,\gamma^6}, \tag{8}$$

we obtain, instead of expressions (7) simpler expressions:

$$w^o = \frac{R}{(2c)^{3/2}Eh}\sum_{n=1}^{\infty} q_n\gamma^{-2}\sin\gamma\beta,\quad T_1^o = -\frac{1}{2\sqrt{2c}}\sum_{n=1}^{\infty} q_n\gamma^{-2}\sin\gamma\beta =$$

$$= -\frac{1}{2\sqrt{2c}}\, q(\beta)\,, \quad G_2^0 = \frac{\sqrt{c}\, R}{2\sqrt{2}} \sum_{n=1}^{\infty} q_n \sin\gamma\beta = \frac{\nu\sqrt{c}\, R}{2\sqrt{2}}\, q(\beta)\,. \tag{9}$$

Here, by virtue of the uniform convergence of series (2), we were able to contract the series in n for force factors to the function $q(\beta)$. In the case of convergence in the mean, we obtain not the function $q(\beta)$, but rather the Dirichlet function, which differs from the function $q(\beta)$ in the values at the discontinuity points.

We will describe the stressed-deformed state of a simple edge effect by an equation generalizing the well-known equation for an axially symmetric edge effect [9]:

$$\frac{\partial^4 w^{ed}}{\partial\alpha^4} + c^{-2} w^{\kappa} = R^4 D^{-1} p(\alpha, \beta)\,, \tag{10}$$

which is written for normal displacement of the shell-panel $w^{ed} = w^{ed}(\alpha, \beta)$. To this equation there correspond the force factors: $T_2^{ed} = -(Eh/R)w^{ed}$, $G_1^{ed} = -(D/R^2)\partial^2 w^{ed}/\partial\alpha^2$, $G_2^{ed} = \nu G_1^{ed}$.

For Eq. (10) it is easy to construct a solution damping with $\alpha \to \pm\infty$ by means of a Fourier cosine transform and then to write integral representations of the desired factors. But there is no need for this, because such solutions are well-known in the literature. So, using the results from [9], we write the values of the force factors on the loading line ($\alpha = 0$):

$$T_2^{ed} = -\frac{1}{2}\, q(\beta) \sqrt[4]{3(1-\nu^2)}/\sqrt{R/h}\,, \quad G_1^{ed} = \frac{1}{4}\, q(\beta) \frac{\sqrt{Rh}}{\sqrt[4]{3(1-\nu^2)}}\,.$$

To calculate the stresses and bending moments, we apply the method of asymptotic synthesis and obtain the formulas

$$T_1(0, \beta) = T_2(0, \beta) = T_1^0 + T_1^{ed} = -\frac{1}{2}\, q(\beta) \sqrt[4]{3(1-\nu^2)}\, \sqrt{R/h}\,,$$

$$G_1(0, \beta) = G_2(0, \beta) = G_1^0 + G_1^{ed} = (1+\nu)\, q(\beta) \sqrt{Rh}\, \left(4\sqrt[4]{3(1-\nu^2)}\right)^{-1}. \tag{11}$$

Analysis of these approximate solutions shows that on the loading line the corresponding force factors are equal to each other and independent of the apex angle of the panel θ; they change along the contour of the shell in the same may as the external load.

For normal displacement of the edge effect we obtain

$$w^{ed}(\alpha, \beta) = \frac{R^3}{\pi D}\, q^*(\beta) \int_0^{\infty} \frac{\cos\alpha\lambda}{\lambda^4 + c^{-2}}\, d\lambda$$

or with allowance for integral (8)

$$w^{ed}(0, \beta) = (2Eh\,(2c)^{1/2})^{-1}\, Rq^*(\beta)\,.$$

Now, from this formula and from the first formula of set (9) we find a total expression for the normal displacement:

$$w(0, \beta) = w^0 + w^{ed} = (2Eh\,(2c)^{1/2})^{-1}\, R \left(q^*(\beta) + c^{-1} \sum_{n=1}^{\infty} q_n \gamma^{-2} \sin\gamma\beta \right). \tag{12}$$

In Eq. (12) $q^*(\beta)$ is a function that is as yet unknown. In [1, 2] this was taken to be either a zeroth harmonic (axially symmetric component) of a series in cosines, or a sum of a series up to the harmonic number n^*. In the case of a panel, the zeroth harmonic is absent. The function $q^*(\beta)$ is to be determined.

In order to estimate the reliability of approximate formulas (11) and (12), we will construct a more strict solution of the problem on the basis of the equations for the moment technical theory of shells, which, as is known, gives good accuracy in determining local stresses [10].

The Donnell–Vlasov equations for a cylindrical panel have the form

$$\nabla^8\Phi + c^{-2}\,\partial^4\Phi/\partial\alpha^4 = R^4 D^{-1} p(\alpha,\beta)\,, \quad w = \nabla^4\Phi\,,$$

$$T_1 = -(Eh/R)\,\partial^4\Phi/\partial\alpha^2\partial\beta^2\,, \quad T_2 = -(Eh/R)\,\partial^4\Phi/\partial\alpha^4\,,$$

$$G_1 = -\frac{D}{R^2}\left(\frac{\partial^2}{\partial\alpha^2} + \nu\frac{\partial^2}{\partial\beta^2}\right)\nabla^4\Phi\,, \quad G_2 = -\frac{D}{R^2}\left(\frac{\partial^2}{\partial\beta^2} + \nu\frac{\partial^2}{\partial\alpha^2}\right)\nabla^4\Phi\,, \quad \nabla^2 = \frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2}. \tag{13}$$

The solution (damping with $\alpha \to \pm\infty$) of the resolving equation given in Eq. (13) can be easily constructed by the Fourier integral method. Using it as the basis and other relations from Eq. (13), we write expressions for the normal displacement and force factors on the loading line:

$$w = \frac{R^3}{\pi D}\sum_{n=1}^{\infty} q_n \sin\gamma\beta \int_0^\infty \frac{(\lambda^2+\gamma^2)^2\,d\lambda}{\Delta(\lambda,\gamma)}\,, \quad T_i = -\frac{EhR^2}{\pi D}\sum_{n=1}^{\infty} q_n \sin\gamma\beta \times$$

$$\times \int_0^\infty \frac{t_i(\lambda,\gamma)\,d\lambda}{\Delta(\lambda,\gamma)}\,, \quad G_i = \frac{R}{\pi}\sum_{n=1}^{\infty} q_n \sin\gamma\beta \int_0^\infty \frac{g_i(\lambda,\gamma)(\lambda^2+\gamma^2)^2}{\Delta(\lambda,\gamma)}\,d\lambda \quad (i = 1, 2)\,,$$

$$\Delta(\lambda,\gamma) = (\lambda^2+\gamma^2)^4 + c^{-2}\lambda^4\,, \quad t_1(\lambda,\gamma) = \gamma^2\lambda^2\,, \quad t_2(\lambda,\gamma) = \lambda^4\,,$$

$$g_1(\lambda,\gamma) = \lambda^2 + \nu\gamma^2\,, \quad g_2(\lambda,\gamma) = \gamma^2 + \nu\lambda^2\,. \tag{14}$$

2. Transformation of Expressions for Force Factors. The improper integrals are calculated by the theory of residues:

$$\int_0^\infty \frac{(\lambda^2+\gamma^2)^2\,d\lambda}{\Delta(\lambda,\gamma)} = \frac{\pi}{4\sqrt{2}\,\gamma^2}\left(\frac{(\gamma^4+\omega)^{1/2}+\gamma^2}{\gamma^4+\omega}\right)^{1/2}, \quad \omega = (16c^2)^{-1}\,,$$

$$\int_0^\infty \frac{\gamma^2\lambda^2\,d\lambda}{\Delta(\lambda,\gamma)} = \int_0^\infty \frac{\lambda^4\,d\lambda}{\Delta(\lambda,\gamma)} = \frac{\pi c}{4\sqrt{2}}\left(\frac{(\gamma^4+\omega)^{1/2}-\gamma^2}{\gamma^4+\omega}\right)^{1/2},$$

$$\int_0^\infty \frac{\lambda^2(\lambda^2+\gamma^2)^2\,d\lambda}{\Delta(\lambda,\gamma)} = \int_0^\infty \frac{\gamma^2(\lambda^2+\gamma^2)^2\,d\lambda}{\Delta(\lambda,\gamma)} = \frac{\pi}{4\sqrt{2}}\left(\frac{(\gamma^4+\omega)^{1/2}+\gamma^2}{\gamma^4+\omega}\right)^{1/2}.$$

Substituting the values of these integrals into Eq. (14), we obtain the following series to determine the forces and moments by the Vlasov–Donnell theory:

$$T_1(0,\beta) = T_2(0,\beta) = -\frac{\sqrt{3(1-\nu^2)}}{2\sqrt{2}}\frac{R}{h}\sum_{n=1}^{\infty}\left(\frac{(\gamma^4+\omega)^{1/2}-\gamma^2}{\gamma^4+\omega}\right)^{1/2} q_n \sin\gamma\beta\,,$$

$$G_1(0,\beta) = G_2(0,\beta) = \frac{(1+\nu)R}{4\sqrt{2}}\sum_{n=1}^{\infty}\left(\frac{(\gamma^4+\omega)^{1/2}+\gamma^2}{\gamma^4+\omega}\right)^{1/2} q_n \sin\gamma\beta\,. \tag{15}$$

Here, just as in the method of asymptotic synthesis, we observe equality of the corresponding force factors, but their distribution over the angular coordinate β differs from the distribution of the external load. If in the radicals the terms γ^4 and γ^2 are neglected in comparison to ω, then expressions (15) go over into Eq. (1). Consequently, the asymptotic synthesis method and the moment technical theory give close results for panels with a large apex angle θ and a large radius-to-thickness ratio R/h. In order to satisfy ourselves of this, we will consider the action on the panel of a radial force P uniformly distributed over the segment $\beta \in [\beta_1; \beta_2]$. In this case, the load density is described by the expression $q(\beta) = P/(\beta_2 - \beta_1)R = \text{const}$ and for calculation of the coefficient q_n we have the relations:

$$q_n = \frac{2P}{\pi R \varphi} \frac{1}{n} \sin \gamma\varphi \sin \gamma\psi , \quad \varphi = \frac{\beta_2 - \beta_1}{2} , \quad \psi = \frac{\beta_1 + \beta_2}{2} .$$

Particular attention should be devoted to the case when the center of the loaded segment is equidistant from the edges of the panel, i.e., $\psi = \theta/2$, since then the greatest stresses and displacements occur. Only odd harmonics remain in series (15). To calculate the force factors at the most stressed point $(0, \theta/2)$, we obtain the following expansions

$$T_1 (0, \theta/2) = T_2 (0, \theta/2) = - \frac{3 (1 - \nu^2) \sqrt{2}\, \theta^2}{4\pi^3 \eta\, h^2} PR \sum_{n=1}^{\infty} \left(\frac{\eta}{\varphi}\right)^3 \left(\frac{(\gamma^4 + \omega)^{1/2} - \gamma^2}{\gamma^4 + \omega}\right)^{1/2} \times \frac{1}{n} \sin \gamma\varphi ;$$

$$G_1 (0, \theta/2) = G_2 (0, \theta/2) = \frac{1 + \nu}{2\pi \sqrt{2}\, \eta} P \sum_{n=1}^{\infty} \frac{\eta}{\varphi} \left(\frac{(\gamma^4 + \omega)^{1/2} + \gamma^2}{\gamma^4 + \omega}\right)^{1/2} \times \frac{1}{n} \sin \gamma\varphi , \tag{16}$$

$$n = 1, 3, 5, ..., \infty , \quad \eta = \pi\varphi/\theta .$$

For calculating stresses and moments at the same point the asymptotic synthesis method yields the closed solutions

$$T_1 \left(0, \frac{\theta}{2}\right) = T_2 \left(0, \frac{\theta}{2}\right) = - \frac{P}{4\varphi \sqrt{Rh}} \sqrt[4]{3 (1 - \nu^2)} ,$$

$$G_1 \left(0, \frac{\theta}{2}\right) = G_2 \left(0, \frac{\theta}{2}\right) = \frac{(1 + \nu) P}{8\varphi \sqrt[4]{3 (1 - \nu^2)}} \sqrt{\left(\frac{h}{R}\right)} . \tag{17}$$

In order to obtain approximate closed solutions on the basis of the equations of moment technical theory, we apply a procedure that accelerate the convergence of series (10) and, therefore, resort to the Clauzen integral [11]

$$\sum_{m=0}^{\infty} \frac{\sin (2m + 1) \eta}{(2m + 1)^2} = - \frac{1}{2} \int_0^{\eta} \ln \left(\tan \frac{t}{2}\right) dt , \quad \eta \geq 0 .$$

Expanding the integrand in powers of t and integrating, we obtain

$$\sum_{m=0}^{\infty} \frac{\sin (2m + 1) \eta}{(2m + 1)^2} \cong \frac{1}{2} \eta \left(\ln \frac{2}{\eta} + 1 - \frac{1}{36} \eta^2 \left(1 + \frac{7}{200} \eta^2 + \frac{31}{17640} \eta^4\right)\right) ,$$

$$\sum_{m=0}^{\infty} \frac{\sin (2m + 1) \eta}{(2m + 1)^4} \cong \eta \left(1.05178 - \frac{1}{12} \eta^2 \left(\ln \frac{2}{\eta} + \frac{11}{6} - \frac{1}{120} \eta^2 - \frac{1}{7200} \eta^4\right)\right) . \tag{18}$$

The accuracy of these formulas decreases with an increase in η. In our case, $0 \leq \eta \leq \pi/2$. When $\eta = \pi/2$, calculations by formulas (18) yield

$$\sum_{m=0}^{\infty} \frac{\sin((2m+1)\pi/2)}{(2m+1)^2} \cong 091607\ (091596)\,,$$

$$\sum_{m=0}^{\infty} \frac{\sin((2m+1)\pi/2)}{(2m+1)^4} \cong 0.98889\ (0.98894)\,.$$

The more accurate values (in brackets) are taken from the tables of the handbook [11]. As we see, in the indicated interval of values of η formulas (18) ensure a good accuracy. Taking them into account, we reduce series (16) to the finite sums

$$T_1\left(0, \frac{\theta}{2}\right) = T_2\left(0, \frac{\theta}{2}\right) = -\frac{3(1-\nu^2)\theta^2}{4\pi^3}\frac{PR}{h^2}\left[\sum_{n=1,3,\dots}^{N}\left(\frac{\sqrt{2}}{\sqrt{\omega}}\frac{\eta^2}{\varphi^3}\left(\frac{(\gamma^4+\omega)^{1/2}-\gamma^2}{\gamma^4+\omega}\right)^{1/2} - \frac{1}{\eta n^3}\right)\times\right.$$

$$\left.\times\frac{1}{n}\sin\gamma\varphi + 1.05178 - \frac{1}{12}\eta^2\left(\ln\frac{2}{\eta} + \frac{11}{6} - \frac{\eta^2}{120} - \frac{\eta^4}{7200}\right)\right];$$

$$G_1\left(0, \frac{\theta}{2}\right) = G_2\left(0, \frac{\theta}{2}\right) = \frac{1+\nu}{2\pi}P\left[\sum_{n=1,3,\dots}^{\infty}\left(\frac{1}{\varphi\sqrt{2}}\left(\frac{(\gamma^4+\omega)^{1/2}+\gamma^2}{\gamma^4+\omega}\right)^{1/2} - \frac{1}{\eta n}\right)\times\right.$$

$$\left.\times\frac{1}{n}\sin\gamma\varphi + \frac{1}{2}\left(\ln\frac{2}{\eta} + 1 - \frac{\eta^2}{36}\left(1 + \frac{7\eta^2}{200} + \frac{31\eta^4}{17640}\right)\right)\right]. \tag{19}$$

Here the upper limit of summation $N = 2\ \mathrm{int}\ (\theta\omega^{1/4}/(2\pi)) + 3$ depends on the relative thickness of the panel and its apex angle. The integral part of $\theta\omega^{1/4}/(2\pi)$ increases with increasing θ and R/h.

The closed solutions constructed for the equations of moment technical theory turned out to be more unwieldy than expressions (17) obtained on the basis of the asymptotic synthesis method.

The results of calculations of the force factors by formulas (16), (17), and (19) are given in Tables 1 and 2. The tables present the dimensionless magnitudes of the force $t_1 = -2\pi h T_1(0, \theta/2)/(\sqrt{3(1-\nu^2)}P)$ and of the moment $g_1 = 4\pi G_1(0, \theta/2)/((1+\nu)P)$. The numbers in Table 1 are obtained at $\nu = 0.3$, $\theta = \pi/2$ and at different values of R/h, $K = \theta/(2\varphi)$. The terms for which $n \le 2000$ were retained in series (16). As K increased, the length of the segment over which load was applied decreased. The case $K = 1$ corresponds to loading of the panel over the entire arc of the directional circumference.

Table 2 contains the values of t_1 and g_1 calculated at $\nu = 0.3$, $R/h = 100$, $\theta = \pi/3$, and $\theta = \pi/4$, i.e., at smaller apex angles than in the first case.

3. Transformation of Expressions for Normal Displacement. To calculate the normal displacement by the moment technical theory, we obtain a series:

$$w(0, \beta) = \frac{R}{(2c)^{3/2}Eh}\sum_{n=1}^{\infty} q_n\gamma^{-2} f(\varepsilon)\sin\gamma\beta\,, \tag{20}$$

in which $f(\varepsilon) = [((1+\varepsilon^2)^{1/2} + \varepsilon)/(1+\varepsilon^2)]^{1/2}$, $\varepsilon = 4\gamma^2 c$.

If we assume that $f(\varepsilon) \equiv 1$, then series (20) coincides with the solution of the Schorer two-term equation. At small values of n the quantity ε is small and $f(\varepsilon)$ is close to unity. Therefore, in the case of low numbers of the harmonics in Eq. (2) the terms of series (7) and (20) for normal displacement differ insignificantly from one another and, consequently, their partial sums are close. As n is increased, when $\varepsilon >> 1$, the function $f(\varepsilon)$ has asymptotics: $f(\varepsilon) \sim \sqrt{2}/\sqrt{\varepsilon} = 1/(\gamma\sqrt{2c})$, i.e., it tends to zero with $n \to \infty$. As we see, the terms of series (7) and (20) have different rates of decrease at infinity, but this does not exert a fundamental influence on the sums of the series in the case of their rapid convergence. The above analysis shows that solution (7) can be brought closer to solution (20) only

TABLE 1. Dimensionless Magnitudes of Force Factors t_1 and g_1 at Different Values of R/h, K, and N

R/h	K	N	Formulas (19)		Formulas (16)		Formulas (17)
			t_1	g_1	t_1	g_1	$t_1 = g_1$
25	1	5	0.323	0.335	0.323	0.335	0.311
	2		0.539	0.703	0.539	0.703	0.622
	3		0.636	1.021	0.636	1.022	0.934
	4		0.685	1.275	0.685	1.275	1.245
	5		0.714	1.482	0.714	1.481	1.556
	7		0.744	1.803	0.744	1.802	2.178
	10		0.764	2.152	0.764	2.150	3.112
100	1	7	0.155	0.155	0.156	0.155	0.156
	2		0.317	0.324	0.317	0.323	0.311
	3		0.446	0.516	0.446	0.517	0.467
	4		0.532	0.703	0.533	0.705	0.622
	5		0.590	0.874	0.590	0.874	0.778
	7		0.659	1.161	0.659	1.158	1.089
	10		0.710	1.490	0.709	1.485	1.556
400	1	11	0.076	0.078	0.078	0.078	0.078
	2		0.153	0.154	0.156	0.155	0.156
	3		0.236	0.236	0.237	0.234	0.233
	4		0.315	0.321	0.317	0.323	0.311
	5		0.385	0.415	0.388	0.419	0.389
	7		0.492	0.612	0.493	0.613	0.545
	10		0.592	0.881	0.590	0.874	0.778

TABLE 2. Dimensionless Magnitudes of Force Factors t_1 and g_1 at Different Values of θ, K, and N

q	K	N	Formulas (19)		Formulas (16)		Formulas (17)
			t_1	g_1	t_1	g_1	$t_1 = g_1$
$\pi/3$	1	5	0.240	0.235	0.241	0.235	0.233
	2		0.446	0.515	0.446	0.515	0.467
	3		0.564	0.790	0.564	0.791	0.700
	4		0.630	1.024	0.630	1.024	0.934
	5		0.670	1.220	0.670	1.219	1.167
	7		0.714	1.533	0.714	1.530	1.634
	10		0.744	1.876	0.743	1.872	2.334
$\pi/4$	1		0.323	0.335	0.323	0.335	0.311
	2		0.539	0.703	0.539	0.703	0.622
	3		0.636	1.021	0.636	1.022	0.934
	4		0.685	1.275	0.685	1.275	1.245
	5		0.714	1.482	0.714	1.481	1.556
	7		0.744	1.803	0.744	1.802	2.178
	10		0.764	2.152	0.764	2.150	3.112

TABLE 3. Values of the Functions $f(\varepsilon_i)$, $f_a(\varepsilon_i)$ at Different Values of ε_i

ε_i	$f(\varepsilon_i)$	$f_a(\varepsilon_i)$	ε_i	$f(\varepsilon_i)$	$f_a(\varepsilon_i)$
0	1.000	1.000	0.6	1.140	1,143
0.1	1.046	1.042	0.7	1.135	1.141
0.2	1.083	1.077	0.8	1.126	1.131
0.3	1.110	1.105	0.9	1.114	1.114
0.4	1.128	1.125	1.0	1.099	1.090
0.5	1.138	1.138			

by refining the terms of series (7) at small values of n. This can be done in the following manner. Let us confine ourselves to the harmonic numbers

$$n \leq N = \text{int}\left(\frac{\theta}{2\pi\sqrt{c}}\right), \tag{21}$$

for which $0 < \varepsilon \leq 1$. In this interval of values of ε the function $f(\varepsilon)$ can be approximated by the expression

$$f_a(\varepsilon) = 1 + 0.46\varepsilon - 0.37\varepsilon^2.$$

The above approximation is obtained by the least-squares method. It gives good accuracy, as is evident from Table 3. The table presents the values of the functions $f(\varepsilon_i)$ and $f_a(\varepsilon_i)$ at the nodal points ε_i. The error remains smaller than 1%.

Replacing $f(\varepsilon)$ by its approximation $f_a(\varepsilon)$, we obtain the following expression for calculating the partial sum of series (20)

$$S(0;\beta) = \sum_{n=1}^{N} q_n\gamma^{-2} f(\varepsilon)\sin\gamma\beta = \sum_{n=1}^{N} q_n\gamma^{-2}\sin\gamma\beta +$$

$$+ 1.84c\sum_{n=1}^{N}\left(1 - \frac{74}{23}c\gamma^2\right) q_n \sin\gamma\beta.$$

Here we can pass from the finite sum to the series only in the first term on the right-hand side of the equation. Having performed this operation, we obtain, instead of Eq. (20), a simplified solution of the Donnell-Vlasov equations:

$$w(0;\beta) = \frac{R}{Eh(2c)^{3/2}}\left(\sum_{n=1}^{\infty} q_n\gamma^{-2}\sin\gamma\beta + 1.84c\omega \times \sum_{n=1}^{N}\left(1 - \frac{74}{23}c\gamma^2\right) q_n\sin\gamma\beta\right), \quad \omega = 1. \tag{22}$$

Comparing it with expression (12), obtained previously by the synthesis method, we find the desired function $q^*(\beta)$, which is

$$q^*(\beta) = 1.84\sum_{n=1}^{N}\left(1 - \frac{74}{23}c\gamma^2\right) q_n\sin\gamma\beta.$$

For this $q^*(\beta)$ function, solution (12) coincides with the simplified solution of the equations in the moment technical theory. But the simplified solution (22) itself will differ insignificantly from the exact one only in the case of rapid convergence of series (20). This is due to the fact that replacement of $f(\varepsilon)$ by $f_a(\varepsilon)$ with a small error is forseen only for $n \leq N$. This condition is met when the coefficients q_n rapidly decrease with an increase in n, which occurs when effects are not highly localized. For highly localized (specifically, concentrated) loadings, solutions (20) and

(22) give close results only at a distance from the zone of loading. In the loading zone itself the series converge slowly and the correction suggested loses meaning. The solution of the two-term equation gives values of displacements close to those from the Donnell–Vlasov equations.

Let us analyze the possibilities of the method of asymptotic synthesis on concrete results of calculations. First we will consider the action of a concentrated force P applied at a point with the coordinates $(0, \psi)$. For this loading

$$q_n = \frac{2P}{R\theta} \sin \gamma\psi \tag{23}$$

and we can sum the series in formula (22). Taking into account that [8] for $\beta \le \psi$

$$\sum_{n=1}^{\infty} n^{-2} \sin \gamma\psi \sin \gamma\beta = \frac{\pi^2}{2} \frac{\beta}{\theta} \left(1 - \frac{\psi}{\theta}\right) \tag{24}$$

while for $\beta \ge \psi$, they should change position in Eq. (24), to calculate the deflections from Eq. (22), we obtain

$$w(0;\beta) = \frac{P}{Eh\,(2c)^{3/2}} \left(\begin{Bmatrix} \beta(1 - \psi/\theta) \\ \psi(1 - \beta/\theta) \end{Bmatrix} + 3.68\omega \frac{c}{\theta} \times \right.$$

$$\left. \times \sum_{n=1}^{N} \left(1 - \frac{74}{23} c\gamma^2\right) \sin \gamma\psi \sin \gamma\beta \right) \quad \text{when} \quad \begin{cases} \beta \le \psi, \\ \beta \ge \psi. \end{cases} \tag{25}$$

Solution (25) is simplified when a concentrated force acts in the middle of the band ($\psi = \theta/2$). It gives a simple formula for calculating maximum deflections under it. Assuming in Eq. (25) that $\omega = 0$, $\beta = \psi = \theta/2$, we find

$$w_{\max} = w\left(0; \frac{\theta}{2}\right) = \frac{P\theta R^{3/2}\,(3\,(1 - \nu^2))^{3/4}}{4Eh^{5/2}}.$$

This compact formula is convenient for estimating the maximum deflections of the panel. It has the same order of accuracy as the Odquist formula [7] for a closed shell.

The calculation of the concentrated loading-induced displacements of the panel by the Donnell–Vlasov theory according to Eqs. (20) and (23) is reduced to the series

$$w(0;\beta) = \frac{2P}{Eh\theta\,(2c)^{3/2}} \sum_{n=1}^{\infty} \gamma^{-2} f(\varepsilon) \sin \gamma\psi \sin \gamma\beta .$$

We simplify for a force acting in the middle of the band. Assuming that $\psi = \theta/2$, we find

$$w(0;t) = \frac{2P}{Eh\theta\,(2c)^{3/2}} \left(\sum_{n=1,3,\dots}^{\infty} \gamma^{-2} \left(f(\varepsilon) - \frac{1}{\gamma\sqrt{2c}}\right) \cos \frac{n\pi t}{\theta} + \right.$$

$$\left. + \left(\frac{\theta}{\pi}\right)^3 \frac{1}{\sqrt{2c}} \sum_{n=1,3,\dots}^{\infty} n^{-3} \cos \frac{n\pi t}{\theta} \right), \quad t = \frac{\theta}{2} - \beta . \tag{26}$$

Here, the asymptotic behavior of the function $f(\varepsilon)$ for $\varepsilon \gg 1$ is taken into account. The first series in Eq. (26) converges very rapidly and in calculating its sum it is sufficient to confine oneself to a small number of initial terms. The second series in Eq. (26) can be approximately summed by integrating the expression [8]

$$\sum_{n=1,3,\ldots}^{\infty} n^{-1} \cos nx = -\frac{1}{2} \ln \left(\tan \frac{x}{2} \right).$$

Expanding ln (tan $x/2$) into series in powers of x and performing double integration, we obtain

$$S(x) = \sum_{n=1,3,\ldots}^{\infty} n^{-3} \cos(nx) = 1.0517998 + \frac{x^2}{4} \left(\ln \frac{x}{2} - \frac{3}{2} + \frac{x^2}{72} \left(1 + \frac{7x^2}{300} + \frac{31x^4}{35280} \right) \right). \tag{27}$$

The accuracy of this formula is worsened with an increase in the variable x. In our case $0 \le x \le \pi/2$. When $x = \pi 2$, it gives $S(\pi/2) = -1.54 \cdot 10^{-5}$ instead of zero. As we can see, in the indicated interval of the values of x the error of the formula is smaller than 10^{-4}. Using Eq. (27) instead of series (26), we obtain the closed approximate solution

$$w(0; t) = \frac{2P}{Eh\theta (2c)^{3/2}} \left(\sum_{n=1,3,\ldots}^{M} \gamma^{-2} \left(f(\varepsilon) - \frac{1}{\gamma \sqrt{2c}} \right) \cos \frac{n\pi t}{\theta} + \right.$$

$$\left. + \left(\frac{\theta}{\pi} \right)^3 \frac{1}{\sqrt{2c}} S\left(\frac{\pi t}{\theta} \right) \right); \quad M = 2 \operatorname{int} \left(\frac{\theta}{4\pi \sqrt{c}} \right) + 3. \tag{28}$$

It turns out to be more complex than expression (25) given by the method of asymptotic synthesis.

Next, using two techniques, we determine the deflections of a panel loaded uniformly by a radial force over a segment of directional circumference. In this case

$$q_n = \frac{2P}{\pi R \varphi n} \sin \gamma\varphi \sin \gamma\psi; \quad \varphi = \frac{\beta_2 - \beta_1}{2}; \quad \psi = \frac{\beta_1 + \beta_2}{2} \tag{29}$$

and the sum of the series in Eq. (22) can be expressed, as before, in elementary functions. To calculate deflections at the center of the loading segment, i.e., at $\beta = \psi$, we obtain the formula

$$w(0; \psi) = \frac{P}{Eh (2c)^{3/2}} \left(\psi \left(1 - \frac{\psi}{\theta} \right) - \frac{1}{4} \varphi + 3.68\omega \frac{c}{\pi\varphi} \times \sum_{n=1}^{N} \left(1 - \frac{74}{23} c\gamma^2 \right) \frac{1}{n} \sin^2 \gamma\psi \sin \gamma\varphi \right), \quad \omega = 1.$$

This formula is simplified in the case of symmetrtic loading, when $\psi = \theta/2$, and takes the form

$$w\left(0; \frac{\theta}{2}\right) = \frac{P}{Eh (2c)^{3/2}} \left(\frac{1}{4} (\theta - \varphi) + 3.68\omega \frac{c}{\pi\varphi} \times \sum_{n=1,3,\ldots}^{\infty} \left(1 - \frac{74}{23} c\gamma^2 \right) \frac{1}{n} \sin \gamma\varphi \right), \quad \omega = 1. \tag{30}$$

Let us analyze what the moment technical theory gives in this case. Substituting the values of q_n from Eq. (29) into Eq. (20), at $\beta = \psi = \theta/2$ we find

$$w\left(0; \frac{\theta}{2}\right) = \frac{P}{Eh \pi\varphi (2c)^{3/2}} \left(\sum_{n=1,3,\ldots}^{\infty} \gamma^{-2} \left(f(\varepsilon) - \frac{1}{\gamma \sqrt{2c}} \right) \frac{1}{n} \times \right.$$

$$\left. \times \sin \gamma\varphi + \left(\frac{\theta}{\pi} \right)^3 \frac{1}{\sqrt{2c}} \sum_{n=1,3,\ldots}^{\infty} n^{-4} \sin \gamma\varphi \right). \tag{31}$$

Next, we will pass from the series to finite sums. Integrating expression (27), we find

$$F(x) = \sum_{n=1,3,\ldots}^{\infty} n^{-4} \sin nx = x \left(1.0517998 + \frac{x^2}{12} \left(\ln \frac{x}{2} - \frac{11}{6} + \frac{x^2}{120} \left(1 + \frac{1}{60} x^2 + \frac{31}{63504} x^4 \right) \right) \right).$$

TABLE 4. Dimensionless Magnitudes of the Normal Displacement $w_1(0; t)$, ..., $w_4(0; t)$ at Different Values of R/h and t

R/h	t	$w_1(0; t)$	$w_2(0; t)$	$w_3(0; t)$	$w_4(0; t)$
	0	104.3	110.6	104.4	104.4
25	$\theta/6$	69.5	75.0	76.8	76.8
	$\theta/3$	34.8	37.9	37.8	37.8
	0	834.0	858.3	834.0	834.2
100	$\theta/6$	556.0	568.6	567.5	567.5
	$\theta/3$	278.0	275.5	277.1	276.9
	0	6672.2	6774.4	6671.8	6676.4
400	$\theta/6$	4448.1	4440.6	4444.7	4446.3
	$\theta/3$	2224.0	2224.4	2224.2	2224.7

TABLE 5. Dimensionless Magnitudes of the Normal Displacement w_5, ... , w_8 at Different Values of R/h and K

R/h	K	$w_5(0; \theta/2)$	$w_6(0; \theta/2)$	$w_7(0; \theta/2)$	$w_8(0; \theta/2)$
	1	52.1	56.8	56.5	56.5
	3	86.9	91.0	93.2	93.2
25	5	93.8	97.5	99.3	99.3
	7	96.8	100.3	101.4	101.4
	10	99.0	102.4	102.7	102.7
	1	417.0	424.2	425.2	425.2
	3	695.0	715.2	717.4	717.4
100	5	750.6	773.5	775.8	775.8
	7	774.5	798.2	798.5	798.6
	10	792.3	816.5	813.5	813.6
	1	3336.1	3351.2	3352.5	3352.5
	3	5560.2	5607.0	5610.7	5610.9
400	5	6005.0	6078.9	6085.5	6084.7
	7	6195.7	6281.8	6290.9	6290.9
	10	6338.6	6432.5	6439.3	6440.7

The error of this formula does not exceed 10^{-5}. Taking this error into account, we transform Eq. (31) into the approximate closed solution

$$w\left(0; \frac{\theta}{2}\right) = \frac{2P}{Eh\,\pi\varphi\,(2c)^{3/2}}\left(\sum_{n=1,3,\ldots}^{\infty} \gamma^{-2}\left(f(\varepsilon) - \frac{1}{\gamma\sqrt{2c}}\right)\frac{1}{n} \times \sin\gamma\varphi + \left(\frac{\theta}{\pi}\right)^{3}\frac{1}{\sqrt{2c}}F\left(\frac{\pi\varphi}{\theta}\right)\right). \qquad (32)$$

Now we will analyze the results of calculations presented in Table 4 and 5. They were obtained at $\nu = 0.3$ and $\theta = \pi/2$. Table 4 contains dimensionless values of the displacements $w_i(0; t) = Ehw(0; t)/P$, $i = \overline{1; 4}$ calculated for different R/h and t. The calculation by formula (25) corresponds to $i = \overline{1; 2}$. When calculating the function $w_1(0; t)$, we assumed that in it $\omega = 0$, i.e., we took only the solution of the Schorer two-term equation. When we calculated the function $w_2(0; t)$, we allowed also for the second term. This corresponds to the corrected solution of the two-term equation based on the asymptotic synthesis method. The values of $w_3(0; t)$ and $w_4(0; t)$ were obtained from formulas (26) and (28). In this case in the series we retained the terms for which $n \le 201$. A comparison of the values of $w_i(0; t)$ shows that the use of asymptotic synthesis for a concentrated force gives good results only far from the point of application of the external load. Deflections of a panel under a force are represented with high accuracy by a closed formula obtained by solving the two-term equation. Within the framework of the Donnell–Vlasov theory, the closed solution (28) gives high accuracy in a wide range of the values of R/h and t. It allows us to avoid numerical summation of series (26). Table 5 contains dimensionless values of the displacements $w_j(0;\ \theta/2) = Ehw(0;\ \theta/2)$; $j = \overline{5; 8}$. They were calculated at the center of the loading segment for different ratios R/h and for $K = \theta/(2\varphi)$. The calculation by formula (30) corresponds to the values of the subscript $j = \overline{5; 6}$. To calculate $w_5(0;\ \theta/2)$ in this formula, we assumed that $\omega = 0$, i.e., we took only the Schorer solution. The function $w_6(0;\ \theta/2)$ was calculated with allowance for the contribution made by solution of the equation of a simple edge effect. The displacements $w_7(0;\ \theta/2)$ and $w_8(0;\ \theta/2)$ were calculated from formulas (31) and (32). In summing up the series in Eq. (31), we retained the terms for which $n \leq 201$.

Conclusion. The analysis of the results, presented in Table 1-5 for force factors and normal displacement, shows that the error of approximate formulas (19) is small in the entire considered range of the change in the parameters R/h and K. The error decreases with a decrease in the length of the loaded segment. An opposite tendency is observed in the method of asymptotic synthesis of a stressed state, namely, the accuracy of formulas (17) depends substantially on the values of R/h and K. The error does not exceed 15% for stresses when $K \leq \theta\sqrt{R/h}\ \theta/4$, or for bending moments when $K \leq \sqrt{R/h}\ \theta/3$. Thus, in the case of nonclosed shells too it is possible to use the simple formulas given by the asymptotic synthesis method for calculations over a certain range of parameters. This method brings the displacements calculated on the basis of the Schorer two-term equation closer to those obtained from the Vlasov–Donnell theory. The closed solution (32) is rather exact and allows one to avoid numerical summation of series (31). According to the Schorer theory, the transition from a concentrated force to loading distributed uniformly over the entire width of the band decreases the normal displacement by a factor of two. Deviations from this relation are also insignificant in calculations of displacements on the basis of the Donnell–Vlasov theory.

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NOTATION

R, h, radius and thickness of shell-panel; E, ν, elasticity modulus and Poisson coefficient of material; p, q, surface and linear loadings; P, force on portion of surface; θ, apex angle of shell-panel; α, β, dimensionless coordinates; β_1, β_2, angular coordinates limiting loaded region; Φ, resolving function of problem; w, normal displacement; T_1, G_1, longitudinal stress and bending moment; T_2, G_2, annular stress and bending moment; 0, ed, subscripts indicating the reference of the desired factors to the ground state and to the edge effect, respectively; m, n, integer-valued parameters of summation; a, approximation.

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